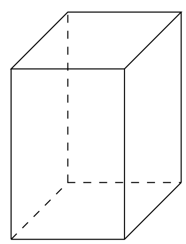
Snow College Mathematics Contest

$\boxed{\text{key}}$

March 19, 2024 **Senior Division: Grades 10-12** **Form: $\boxed{\text{T}}$**

Bubble in clearly the single best choice for each question you choose to answer.

1. For a science project, you decide to create a sky lantern. A wire frame is built with a square base as shown below. Before you open it, you measure the width of the garbage bag that you will use to be 40 in while still closed. You open the bag and pull it over the frame so the opening of the bag fits snugly at the square base. What is the length of the diagonals of the square base?



$\boxed{\text{(A)}}$ 28.3 in
(B) 19.3 in
(C) 26.9 in
(D) 30.4 in
(E) 54.4 in

$\boxed{\text{SOLN}}$ Each side of the square will be $\frac{40}{2}$ in so the diagonal will be $\sqrt{2}(20\,\text{in})$. $\square$

2. Jane works at a car wash. 7 cars pull up at the same time. 3 are red, 2 are green, and 2 are blue. Her car wash has 2 bays. How many different ways can 2 cars that are the same color be selected to be washed at the same time — one in each bay?

(A) 3
(B) 4
$\boxed{\text{(C)}}$ 5
(D) 6
(E) 7

$\boxed{\text{SOLN}}$ The three red can be washed 3 ways: $\{r_1, r_2; r_1, r_3; r_2, r_3\}$ and one way for green and one way for blue. $\square$

3. If you flip a coin 3 times and then repeat the experiment, what is the probability of getting exactly 2 heads out of 3 both times?

(A) $\frac{2}{3}$
(B) $\frac{15}{64}$
(C) $\frac{6}{64}$
$\boxed{\text{(D)}}$ $\frac{9}{64}$
(E) $\frac{24}{64}$

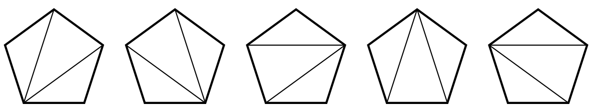
$\boxed{\text{SOLN}}$ The probability of getting exactly 2 heads in three flips is $\frac{3}{8}$. The probability of this happening twice is $\frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}$. $\square$

4. Dunddin Dede does not dislike Dumdums. A letter is drawn from the previous sentence. If it is a vowel, we return the letter and draw again. What is the probability that we end up with the chosen letter being d or D?

$\boxed{\text{(A)}}$ $\frac{9}{20}$
(B) $\frac{9}{31}$
(C) $\frac{1}{12}$
(D) $\frac{1}{8}$
(E) $\frac{8}{31}$

$\boxed{\text{SOLN}}$ There are 9 in 20 consonants. $\square$

5. A polygon with $n + 2$ sides can be cut into n triangles. The number of different ways to cut the polygon into triangles using line segments from one vertex to another generates a sequence called the *Catalan numbers*: 1, 2, 5, ... For example, a square can be cut into two triangles in two different ways. A pentagon can be cut into 3 triangles in five different ways as illustrated.



In how many different ways can a hexagon be cut into 4 triangles?

(A) 7

(B) 11

$\boxed{\text{(C)}}$ 14

(D) 42

(E) 55

$\boxed{\text{SOLN}}$ The Catalan number sequence can be generated by the formula $C(n) = \frac{(2n)!}{n!(n+1)!}$. For a six-sided polygon, this is $C(4) = \frac{8!}{4!5!} = \frac{(8)(7)(6)}{(4)(3)(2)(1)} = 14$.

Alternatively, this can be expressed as a recursion formula with $a_0 = 1$ and $a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}$. □

6. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$, then what is the value of $x + y + z$?

(A) 50

(B) 58

$\boxed{\text{(C)}}$ 89

(D) 111

(E) 1296

$\boxed{\text{SOLN}}$ $\log_2(\log_3(\log_4 x)) = 0$ means $\log_3(\log_4 x)) = 2^0 = 1$ and $\log_4 x = 3^1$ and $x = 4^3 = 64$. Similarly $\log_3(\log_4(\log_2 y)) = 0$ gives $y = 16$ and $\log_4(\log_2(\log_3 z)) = 0$ gives $z = 9$. □

7. An ambitious chef wants to make circular pizzas on her square baking sheet. One large pizza is going to fill as much space as possible. Then a tiny pizza, as large as possible, will be placed in each of the four corners. The baking sheet has sides of 2 ft. What is the area of one of the tiny pizzas in sq. ft?

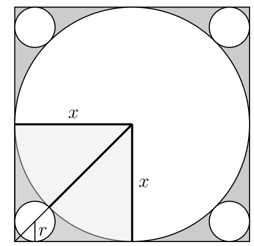
(A) $\pi - \sqrt{2}$

$\boxed{\text{(B)}}$ $\pi(\sqrt{2} - 1)^4$

(C) $\pi(\sqrt{2} - 1)^2$

(D) $4\pi(2 - \sqrt{2})$

(E) $2\pi(2 - \sqrt{2})$



$\boxed{\text{SOLN}}$ Call the radius of the large circle x. Then the diagonal of the shaded square is $\sqrt{2}x$. The distance from the edge of the large circle to the corner will be $\sqrt{2}x - x = (\sqrt{2} - 1)x$. If the radius of a small circle is r, then the same distance is $r + \sqrt{2}r = (\sqrt{2} + 1)r$. Set these two expressions equal: $r(\sqrt{2} + 1) = x(\sqrt{2} - 1)$.

$$r = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} x = \left(\sqrt{2} - 1\right)^2 x$$

The area of a small circle is $A = \pi r^2 = \pi\left(\sqrt{2} - 1\right)^4 x^2$, with $x = 1$ ft. □

8. One side of the gray square is increased by 3 cm while its adjacent side is decreased by 2 cm. The perimeter of the resulting rectangle is 22 cm. What is the area of the original gray square?

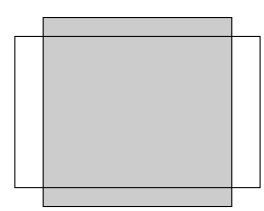
(A) $9\,\text{cm}^2$

(B) $16\,\text{cm}^2$

$\boxed{\text{(C)}}$ $25\,\text{cm}^2$

(D) $64\,\text{cm}^2$

(E) $121\,\text{cm}^2$



$\boxed{\text{SOLN}}$ If x is a side length of the square, the perimeter of the new rectangle will be $22 = 2(x + 3) + 2(x - 2) = 4x + 2$. Solving gives $x = 5$ and the square's area is $25\,\text{cm}^2$. □

9. Let a_n be the remainder of p^n divided by q. Note that a_n will be a repeating pattern for all integer values of p and q. What is the remainder of $(2^{241} - 2)$ divided by 5?

   $\boxed{\text{(A)}}$ 0

   (B) 1

   (C) 2

   (D) 3

   (E) 4

   $\boxed{\textit{SOLN}}$ For $q = 5$ and $p = 2$, the sequence is $2, 4, 3, 1, 2, 4, 3, 1, 2, \ldots$ with a repeating block of size 4. Since 241 is one more than a multiple of 4, $2^{241} \div 5$ has remainder 2. Therefore, $2^{241} - 2$ is an exact multiple of 5. □

10. Find the value of the shaded area assuming that the triangles are equilateral and the pattern continues indefinitely.

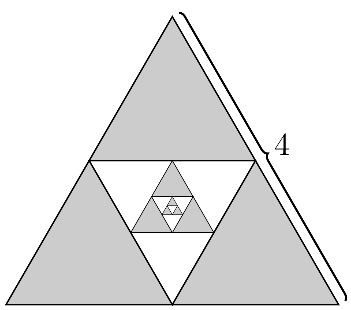
    (A) $\dfrac{4\sqrt{3}}{3}$

    (B) $\dfrac{7\sqrt{3}}{2}$

    (C) $\dfrac{5\sqrt{3}}{4}$

    (D) $\dfrac{16\sqrt{3}}{3}$

    $\boxed{\text{(E)}}$ $\dfrac{16\sqrt{3}}{5}$



    $\boxed{\textit{SOLN}}$ The area of an equilateral triangle with base ℓ is $\dfrac{\ell^2\sqrt{3}}{4}$. The largest triangle shown has area $4\sqrt{3}$. Each subsequent triangle, whether shaded or not, has one-fourth the area. Hence, with adding and subtracting:

$$\text{Total Shaded Area} =$$

$$4\sqrt{3} \cdot \left(1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots\right) =$$

$$\frac{4\sqrt{3}}{1 + \frac{1}{4}} = \frac{16\sqrt{3}}{5}$$ □

11. A basketball player makes free throws 80% of the time. He plans to shoot 5 free throws. What is the probability that he will make exactly 4 of the 5 attempts?

    $\boxed{\text{(A)}}$ 41%

    (B) 50%

    (C) 80%

    (D) 62%

    (E) 33%

    $\boxed{\textit{SOLN}}$

$$\binom{5}{4} \times (0.8)^4 \times (0.2) = (0.8)^4 = 0.4096$$ □

12. In geometry, an *arbelos* is a plane region bounded by three semicircles such that each corner of each semicircle is shared with one of the others, all on the same side of a straight line that contains their diameters. This is the shaded region illustrated below.
    Some remarkable properties of the arbelos are that

    - $BFGD$ is a rectangle
    - line DF is tangent to the smaller semicircles
    - the area of the arbelos equals the area of the circle with diameter BG.

    If $AB = 2$ and $AC = 8$, what is the area of the arbelos in square units?

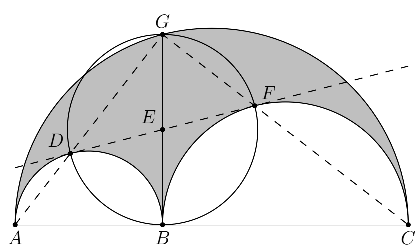
    (A) π

    (B) 2π

    $\boxed{\text{(C)}}$ 3π

    (D) 5π

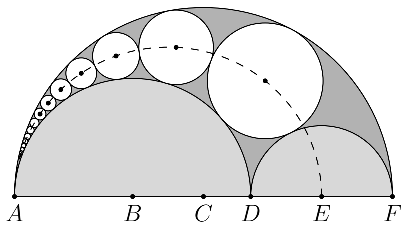
    (E) 10π



    $\boxed{\textit{SOLN}}$ Reflecting the semicircles across $\overline{AC}$, we find that twice the area of the arbelos equals the area of the large circle minus the area of the two smaller circles. The larger circle radius is 4 and the smaller radii are 1 and 3. This gives $2(\text{Area}) = 16\pi - \pi - 9\pi = 6\pi$.
    The first recorded study of this shape is attributed to Archimedes. □

13. A *Pappus chain* is a sequence of circles tangent to the generating circles of an arbelos (dark shading) as illustrated below. The points B, C, and E are the centers of the circles generating the arbelos. The centers of the circles in the chain lie along an ellipse (dashed) with focal points B and C. If $AD = 6$ and $AF = 8$, what is the semi-major axis length of the ellipse?

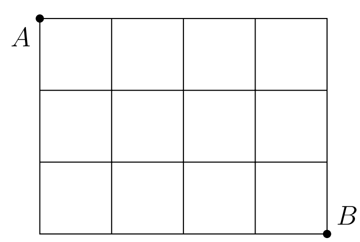
(A) 2.5
(B) 3
$\boxed{\text{(C)}}$ 3.5
(D) 4
(E) 4.5



SOLN $AB = 3$ and $BC = 1$ which makes the focal length $2c = 1$ or $c = 0.5$. The center of the ellipse will be halfway between B and C a distance of 0.5 units from B making the semi-major axis length $a = 3.5$.
Alt. Soln.: Major axis $= AE = 7 \implies$ semi-major axis $= 7/2$. □

14. Going only right or down, how many different ways are there to get from point A (upper left corner) to point B (lower right corner) of the 3×4 grid below?

(A) 28
(B) 32
$\boxed{\text{(C)}}$ 35
(D) 56
(E) 84



SOLN The trip will take exactly 7 steps, 3 of which are down and four are to the right. Since order doesn't matter, this is a combination: ${}_7C_3 \equiv \binom{7}{3} = {}_7C_4 \equiv \binom{7}{4} = 35$. One can also note the entries of Pascal's triangle at each corner. □

15. How many pairs of integers a and b are solutions to the equation?

$$a(a+1)(a+2) = b^2 + 4$$

Hint: use divisibility rules.

$\boxed{\text{(A)}}$ 0
(B) 1
(C) 2
(D) 3
(E) ∞

SOLN The left side must be divisible by 2 and 3; the right side will also always be even, but it is never divisible by 3. From `https://tinyurl.com/ycxhjk2n` □

16. One extension of the real numbers is the complex numbers $a + b\mathrm{i}$, where a, b are real numbers and $\mathrm{i}^2 = -1$. Another is the dual numbers $a + b\varepsilon$, where a, b are real numbers and $\varepsilon^2 = 0$. Use the same technique you use to divide complex numbers to compute $(4 + 3\varepsilon) \div (2 + 5\varepsilon)$.

(A) $\frac{5}{2} - \frac{3}{4}\varepsilon$
(B) $3 + 2\varepsilon$
(C) $\frac{5}{3} - 4\varepsilon$
(D) $\frac{4}{5} + 3\varepsilon$
$\boxed{\text{(E)}}$ $2 - \frac{7}{2}\varepsilon$

SOLN

$$\frac{(4+3\varepsilon)}{(2+5\varepsilon)}\frac{(2-5\varepsilon)}{(2-5\varepsilon)} = \frac{8-14\varepsilon}{4} = 2 - \frac{7}{2}\varepsilon$$

□

17. Simplify.
$$\sqrt{\mathrm{i}\sqrt{\mathrm{i}\sqrt{\mathrm{i}\ldots}}}$$

(A) $\sqrt{2\mathrm{i}}$

(B) $\mathrm{i}\sqrt{\mathrm{i}}+\mathrm{i}$

(C) $-\sqrt{2\mathrm{i}}$

(D) $-2\mathrm{i}$

$\boxed{\text{(E)}}$ i

$\boxed{\text{SOLN}}$ Method 1: Set $x = \sqrt{\mathrm{i}\sqrt{\mathrm{i}\sqrt{\mathrm{i}\ldots}}}$. Then $x = \sqrt{\mathrm{i}x}$ and $x^2 = \mathrm{i}x \implies x = \mathrm{i}$.
Method 2: $\sqrt{\mathrm{i}\sqrt{\mathrm{i}\sqrt{\mathrm{i}\ldots}}} = \sqrt{\mathrm{i}}\sqrt{\sqrt{\mathrm{i}}}\sqrt{\sqrt{\sqrt{\mathrm{i}}}}\ldots = \left(\mathrm{i}^{\frac{1}{2}}\right)\left(\mathrm{i}^{\frac{1}{4}}\right)\left(\mathrm{i}^{\frac{1}{8}}\right)\ldots = \mathrm{i}^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)} = \mathrm{i}^1$ (Principal value)
Method 3: Use the polar form: $\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$. Then $\sqrt{\mathrm{i}} = \mathrm{e}^{(\mathrm{i}\frac{\pi}{4})}$ and $\sqrt{\sqrt{\mathrm{i}}} = \mathrm{e}^{(\mathrm{i}\frac{\pi}{8})}\ldots$ $\sqrt{\mathrm{i}}\sqrt{\sqrt{\mathrm{i}}}\sqrt{\sqrt{\sqrt{\mathrm{i}}}}\ldots = \mathrm{e}^{(\mathrm{i}\frac{\pi}{4})}\mathrm{e}^{(\mathrm{i}\frac{\pi}{8})}\mathrm{e}^{(\mathrm{i}\frac{\pi}{16})}\ldots = \mathrm{e}^{\left(\mathrm{i}\frac{\pi}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)\right)} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$. But Method 3 can find additional values for the expression because the complex exponential is a multi-valued function. Pursue $\mathrm{i} = \mathrm{e}^{(\mathrm{i}\frac{\pi}{2}+2\pi n)}$ where $n \in \mathbb{Z}$ to its conclusion. `https://www.youtube.com/watch?v=WfKR_MYu_UA` □

18. If $y = u^3 + u$ and $u = x^2 - 1$, find $\frac{dy}{dx}$.

(A) $6x^5 + 2x$

(B) $x^3 - 2$

(C) $3x^5 + 12x^3$

(D) $8x^5 + 12x$

$\boxed{\text{(E)}}$ $6x^5 - 12x^3 + 8x$

$\boxed{\text{SOLN}}$ Use the Chain Rule. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2+1)(2x)$. Substitution for u gives $\frac{dy}{dx} = (3(x^2-1)^2+1)(2x) = 6x^5 - 12x^3 + 8x$. □

19. Compute the limit (if it exists).
$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)$$

(A) 0

(B) 1/2

$\boxed{\text{(C)}}$ 1

(D) 3/2

(E) The limit does not exist.

$\boxed{\text{SOLN}}$ Multiplying the top and bottom by $(\sqrt{x^2+x+1} + \sqrt{x^2-x})$ simplifies it to $\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$ Then divide everything by x to get $\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{2} = 1$ □

20. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

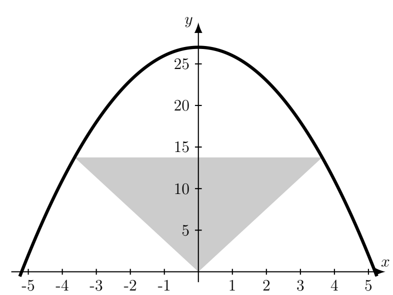
(A) $20\sqrt{3}$

(B) $\frac{45\sqrt{3}}{4}$

(C) 48

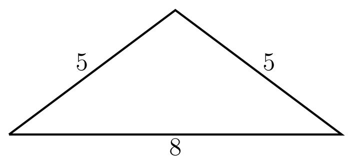
$\boxed{\text{(D)}}$ 54

(E) 60



$\boxed{\text{SOLN}}$ The triangle area is $A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2x)(y) = 27x - x^3$. The derivative is $A' = 27 - 3x^2$ which leads to a critical value of $x = 3$. At this maximum, the area is $A = 27(3) - 3^3 = 81 - 27 = 54$. □

21. What is the area of the triangle?



$\boxed{\text{(A)}}$ 12
(B) 12.5
(C) 20
(D) 25
(E) 40

SOLN The short way is Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is half of the perimeter and $a, b, \&\ c$ are the side lengths.

$$\sqrt{9(9-5)(9-5)(9-8)} =$$
$$= \sqrt{(9)(4)(4)(1)} = (3)(4) = 12$$

Alternatively, drop a vertical from the top creating two 3-4-5 right triangles, each of which has $B = 4$, $H = 3$. □

22. What is the sum of all odd numbers from 1 to 1999? (Hint: $1+3=4$, $1+3+5=9$, $1+3+5+7=16$.)

(A) 999 500
$\boxed{\text{(B)}}$ 1 000 000
(C) 1 999 000
(D) 3 996 001
(E) 4 000 000

SOLN The hint shows that the sum of the first n odd numbers is n^2. The proof is not hard, but omitted. So the sum of the odd numbers from 1 to 1999 is the square of the number of odd numbers in that set. By inspection, there are 1000 odd numbers in that range, so the sum of these is $1000^2 = 1,000,000$. □

23. How many solutions does the system of equations have?

$$\begin{cases} x + 3y = 0 \\ 2x - 3y = 9 \end{cases}$$

(A) 0
$\boxed{\text{(B)}}$ 1
(C) 2
(D) 3
(E) infinitely many

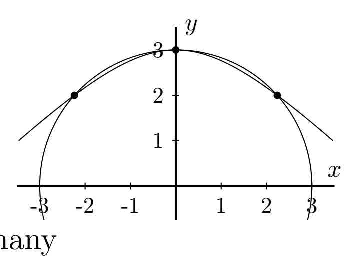
SOLN This linear system corresponds to

$$\begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$$

The determinant of this matrix is $1(-3) - (3)(2) = -9 \neq 0$, so there is exactly one solution to the system. Second method: the equations are lines with different slopes, so they must intersect exactly once. Note: neither method immediately provides the solution. □

24. How many solutions does the system of equations have?

$$\begin{cases} x^2 + y^2 = 9 \\ -x^2 + (y-5)^2 = 4 \end{cases}$$



(A) 0
(B) 1
(C) 2
$\boxed{\text{(D)}}$ 3
(E) infinitely many

SOLN The first equation is a circle, the second a hyperbola whose upper branch crosses the circle at $(\pm\sqrt{5}, 2)$ and kisses (osculates) the circle at $(0, 3)$.

□

25. What is the sum of the roots of the following polynomial?

$$x^6-8x^5-51x^4+302x^3+260x^2-1944x+1440$$

(A) -1994

(B) -12

(C) -8

(D) 4

$\boxed{\text{(E)}}$ 8

$\boxed{\mathcal{SOLN}}$ The sum of roots of a polynomial is the negative of the coefficient of the second term of the polynomial, when the terms are written in descending order by exponent, assuming the leading coefficient is 1. The second coefficient is -8, so the sum of the roots is 8. □

26. Three disks of equal radius are mutually tangent as in the figure below. A rubber band is wrapped around the outside of the group. The distance from the base to the top is exactly 1. Find the total length of the band.

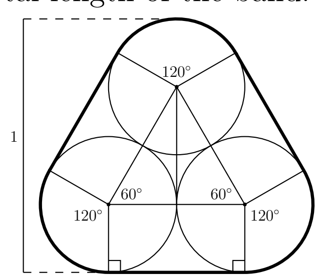
(A) $2\pi+\sqrt{3}$

(B) 3π

(C) $\frac{3+2\pi}{\pi-\sqrt{3}}$

$\boxed{\text{(D)}}$ $\frac{6+2\pi}{2+\sqrt{3}}$

(E) 6π



$\boxed{\mathcal{SOLN}}$ Connecting the centers of the circle forms an equilateral triangle with side lengths of $2r$. The altitude of this triangle will be $\sqrt{3}r$ and thus $2r+\sqrt{3}r=1$ or $r=\frac{1}{2+\sqrt{3}}$. Connecting the centers to the points of tangency shows that the portion of each disk where the band touches has a central angle of 120°. The band covers three of these sections or 360° which has a circumference of $2\pi r$. Combine this with the three segments of length $2r$ and the total length is $(6+2\pi)r=\frac{6+2\pi}{2+\sqrt{3}}$. □

27. A random sample of 100 students polled about pizzas they like gave these results:

- 11 picked only ham & pineapple
- 32 picked only pepperoni
- 23 picked only supreme
- 8 picked ham & pineapple as well as pepperoni
- 10 picked pepperoni as well as supreme
- 7 picked ham & pineapple as well as supreme
- 9 picked all three types of pizza

What is the probability that a randomly chosen student likes ham & pineapple?

$\boxed{\text{(A)}}$ 0.35

(B) 0.11

(C) 0.07

(D) 0.09

(E) 0.54

$\boxed{\mathcal{SOLN}}$ $(11+8+7+9)/100=0.35$ □

28. Three friends (Larry, Moe, Curly) went to Shemp's Grill together. They each ordered a different entreé (cheeseburger, chicken nuggets, or a salad) and a different flavor (vanilla, chocolate, or strawberry) of ice cream. The clues below help determine who ordered which entreé and ice cream flavor.

- The three friends are Curly, the one who ordered a cheeseburger, and the one who ordered chocolate ice cream.
- Moe did not order a salad.
- The chicken nuggets were ordered by either Larry or the one who ordered the chocolate ice cream (but not both).
- The one who ordered a salad does not like vanilla ice cream.

Which of the statements is true?

(A) Moe ordered vanilla ice cream.

(B) Curly ordered chicken nuggets.

(C) Curly ordered vanilla ice cream.

$\boxed{\text{(D)}}$ Larry ordered a cheeseburger.

(E) Larry ordered strawberry ice cream.

$\boxed{\mathcal{SOLN}}$ Curly did not order the cheeseburger or the chocolate ice cream. Larry also did not order chocolate ice cream. Thus, Moe ordered the chocolate ice cream. As the one who ordered chocolate ice cream is not the one who ordered the cheeseburger, Larry must be the one who ordered the cheeseburger. □

29. Jen, Connor, and Bob are each collecting lapel pins. Bob has twice as many pins as Jen. Connor has three fewer pins than Bob. Jen's pins are eight more than one-third of Connor's pins. What is the total number of pins in the three collections combined?

(A) 95

(B) 39

(C) 63

(D) 81

$\boxed{\text{(E)}}$ 102

$\boxed{\mathcal{SOLN}}$ $b = 2j,\ c = b - 3,\ j = 8 + \frac{1}{3}c \implies c = 39,\ b = 42,\ j = 21$ □

30. Integer n is a solution to $8^n+4^n+2^n+1 = 585$. What is the value of $\sin\left(\frac{n\pi}{3}\right)$?

(A) -1

(B) $-\frac{\sqrt{2}}{2}$

(C) $-\frac{1}{2}$

$\boxed{\text{(D)}}$ 0

(E) $\frac{\sqrt{3}}{2}$

$\boxed{\mathcal{SOLN}}$ $8^n + 4^n + 2^n + 1 = 585 \implies n = 3$ (by trial and error). $\sin\left(\frac{n\pi}{3}\right) = \sin(\pi)$ □

31. Suppose that a dataset has a mean of x and a standard deviation of y. We transform the dataset by multiplying each value by 3 and adding 4. Ex: 5 would be transformed to $3 \cdot 5 + 4 = 19$. What are the mean and standard deviation of the new dataset?

(A) x, y

(B) $3x, 4y$

(C) $4x, 3y$

(D) $(3x + 4), (3y + 4)$

$\boxed{\text{(E)}}$ $(3x + 4), 3y$

$\boxed{\mathcal{SOLN}}$ The mean is a measure of the center, so the new mean will be $3x + 4$, and the standard deviation will be $3y$. □

32.

$$\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) =$$

(A) 0

$\boxed{\text{(B)}}$ $\frac{\pi}{4}$

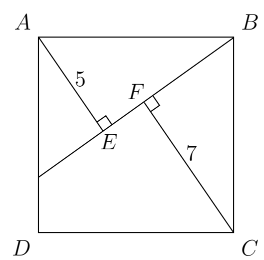
(C) $-\frac{\pi}{4}$

(D) $\frac{3\pi}{4}$

(E) 1

$\boxed{\mathcal{SOLN}}$ $\frac{\pi}{4}$ because of the range of the arcsin function. □

33. Let $ABCD$ be a square, and l be a line segment from B to a point on side AD. If A is 5 cm from l and C is 7 cm from l, find the area of $ABCD$.



$\boxed{\text{(A)}}$ 74

(B) 76

(C) 72

(D) 81

(E) 64

$\boxed{\mathcal{SOLN}}$ Let E be the point on l closest to A and F be the point on l closest to B. Triangle ABE is congruent to BCF. Letting $AB = BC = s$ and using the Pythagorean theorem, we find that $s^2 = 5^2 + 7^2 = 74$. □

34. What is the binary representation of the base ten number 2024?

$\boxed{\text{(A)}}$ 11111101000

(B) 111111001100

(C) 11110110000

(D) 11111010100

(E) 11110110100

$\boxed{\mathcal{SOLN}}$ The algorithm is to repeatedly divide by two and collect the remainders. Or by elimination: because the prime factorization of 2024 ($= 2^3 \cdot 11 \cdot 23$) has exactly three 2s, the binary representation must end in exactly three 0s. □

35. A *proper factor* of a positive integer is any factor of that number, excluding the number itself. For example, the proper factors of 9 are 1 and 3. An *abundant number* is a whole number for which the sum of its proper factors is greater than the number itself. What is the sum of the two smallest abundant numbers?

(A) 24

$\boxed{\text{(B)}}$ 30

(C) 32

(D) 38

(E) 40

$\boxed{\mathcal{SOLN}}$ The first two abundant numbers are 12 ($6 + 4 + 3 + 2 + 1 = 15$) and 18 ($9 + 6 + 3 + 2 + 1 = 21$) which sum to 30 (which is itself an abundant number). Additional note: the smallest odd abundant number is 945 and the smallest abundant number not divisible by 2 or 3 is 5391411025 with distinct prime factors 5, 7, 11, 13, 17, 19, 23, 29. □

36. A *Collatz sequence* starts with a positive integer n. If n is odd, then the next number in the sequence is $3n + 1$, but if n is even then perform $n/2$. Repeat the rule with each result. An as-yet-unproven, but so-far-un-excepted, conjecture is that all such sequences eventually arrive at 1. The number of steps required is not easily predictable. For example the sequence starting with 9 is 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. The assumption that larger numbers have longer sequences is not always true. Which of the numbers below has a longer sequence than for $n = 9$?

(A) 11

(B) 13

(C) 17

$\boxed{\text{(D)}}$ 19

(E) 21

$\boxed{\text{SOLN}}$ The first three choices are quickly eliminated since they show up in the sequence for 9. 21 can also be eliminated because it's sequence is 21, 64, 32, 16, 8, 4, 2, 1. $n = 19$ yields 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. With five numbers preceding 22, this is one longer than the sequence starting with 9. □

37. Given three positive integers, x, y, z, consider the question: Is $x - y$ odd? Possibly useful information:

- Statement 1: $x = z^2$
- Statement 2: $y = (z - 1)^2$

What is true about our ability to answer the question using statements 1 and/or 2?

(A) Only the information in statement 1, taken alone, is sufficient.

(B) Only the information in statement 2, taken alone, is sufficient.

(C) The information in either statement 1 or 2, taken alone, is sufficient.

$\boxed{\text{(D)}}$ The information in statements 1 & 2, taken together, is sufficient.

(E) Even taking both statements together, there is insufficient information to answer the question.

$\boxed{\text{SOLN}}$ $z^2 - (z - 1)^2 = 2z - 1$, which is odd. Either statement alone is insufficient. □

38. Compute the radius of the circle with equation $x^2 + 12x + y^2 + 4y = -4$.

(A) 5.5

$\boxed{\text{(B)}}$ 6

(C) 7

(D) $7\sqrt{2}$

(E) 8

$\boxed{\text{SOLN}}$ Completing the square in x and y gives $x^2 + 12x + 36 + y^2 + 4y + 4 = -4 + 36 + 4$ so $r^2 = 36$ and $r = 6$. □

39. For the function $f(x) = x^2 + 2x - 5$, compute the value of $f(f(f(1)))$.

(A) -5

(B) 5

$\boxed{\text{(C)}}$ 10

(D) 12

(E) 115

$\boxed{\mathcal{SOLN}}$ $f(1) = 1^2 + 2(1) - 5 = -2$ and $f(-2) = (-2)^2 + 2(-2) - 5 = -5$ and $f(-5) = (-5)^2 + 2(-5) - 5 = 10$. $\square$

40. Find the intersection point of the diagonals of the parallelogram $ABCD$ for $A(2, -1)$, $B(5, 2)$, $C(7, -3)$, and $D(4, -6)$.

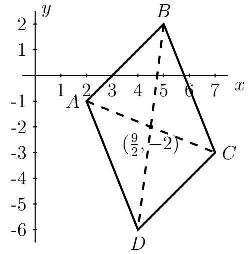
(A) $(\frac{9}{2}, -\frac{5}{2})$

(B) $(4, -2)$

(C) $(5, -3)$

(D) $(\frac{9}{2}, -3)$

$\boxed{\text{(E)}}$ $(\frac{9}{2}, -2)$



$\boxed{\mathcal{SOLN}}$ The diagonals of a parallelogram bisect each other. Find the midpoint between opposite vertices.

$$\left(\frac{2+7}{2}, \frac{-1-3}{2}\right)$$

$\square$